EIGENVALUE COINCIDENCES AND K-ORBITS, I

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Abstract. We study the variety $\mathfrak{g}(l)$ consisting of matrices $x \in \mathfrak{gl}(n,\mathbb{C})$ such that x and its $n-1$ by $n-1$ cutoff x_{n-1} share exactly l eigenvalues, counted with multiplicity. We determine the irreducible components of $\mathfrak{g}(l)$ by using the orbits of $GL(n-1,\mathbb{C})$ on the flag variety $\mathcal{B}$ of $\mathfrak{gl}(n,\mathbb{C})$. More precisely, let $\mathfrak{b} \in \mathcal{B}$ be a Borel subalgebra such that the orbit $GL(n-1,\mathbb{C}) \cdot \mathfrak{b}$ in $\mathcal{B}$ has codimension l. Then we show that the set $Y_{\mathfrak{b}} := \{\mathrm{Ad}(g)(x) : x \in \mathfrak{b} \cap \mathfrak{g}(l), g \in GL(n-1,\mathbb{C})\}$ is an irreducible component of $\mathfrak{g}(l)$, and every irreducible component of $\mathfrak{g}(l)$ is of the form $Y_{\mathfrak{b}}$, where $\mathfrak{b}$ lies in a $GL(n-1,\mathbb{C})$-orbit of codimension l. An important ingredient in our proof is the flatness of a variant of a morphism considered by Kostant and Wallach, and we prove this flatness assertion using ideas from symplectic geometry.

1. Introduction

Let $\mathfrak{g} := \mathfrak{gl}(n,\mathbb{C})$ be the Lie algebra of $n \times n$ complex matrices. For $x \in \mathfrak{g}$, let $x_{n-1} \in \mathfrak{gl}(n-1,\mathbb{C})$ be the upper left-hand $n-1$ by $n-1$ corner of the matrix x. For $0 \leq l \leq n-1$, we consider the subset $\mathfrak{g}(l)$ consisting of elements $x \in \mathfrak{g}$ such that x and x_{n-1} share exactly l eigenvalues, counted with multiplicity. In this paper, we study the algebraic geometry of the set $\mathfrak{g}(l)$ using the orbits of $GL(n-1,\mathbb{C}) \times GL(1,\mathbb{C})$ on the flag variety $\mathcal{B}$ of Borel subalgebras of $\mathfrak{g}$. In particular, we determine the irreducible components of $\mathfrak{g}(l)$ and use this to describe elements of $\mathfrak{g}(l)$ up to $GL(n-1,\mathbb{C}) \times GL(1,\mathbb{C})$-conjugacy.

In more detail, let $G = GL(n,\mathbb{C})$ and let $\theta : G \to G$ be the involution $\theta(x) = dxd^{-1}$, where $d = \mathrm{diag}[1,\dots,1,-1]$. Let $K := G^{\theta} = GL(n-1,\mathbb{C}) \times GL(1,\mathbb{C})$. It is well-known that K has exactly n closed orbits on the flag variety $\mathcal{B}$, and each of these closed orbits is isomorphic to the flag variety $\mathcal{B}_{n-1}$ of Borel subalgebras of $\mathfrak{gl}(n-1,\mathbb{C})$. Further, there are finitely many K-orbits on $\mathcal{B}$, and for each of these K-orbits Q, we consider its length $l(Q) = \dim(Q) - \dim(\mathcal{B}_{n-1})$. It is elementary to verify that $0 \leq l(Q) \leq n-1$. For $Q = K \cdot \mathfrak{b}_Q$, we consider the K-saturation $Y_Q := \mathrm{Ad}(K)\mathfrak{b}_Q$ of $\mathfrak{b}_Q$, which is independent of the choice of $\mathfrak{b}_Q \in Q$.

Theorem 1.1. *The irreducible component decomposition of $\mathfrak{g}(l)$ is*

$$\mathfrak{g}(l) = \bigcup_{l(Q)=n-1-l} Y_Q \cap \mathfrak{g}(l). \tag{1.1}$$

The proof uses several ingredients. The first is the flatness of a variant of a morphism studied by Kostant and Wallach [KW06], which implies that $\mathfrak{g}(l)$ is equidimensional. We

prove the flatness assertion using dimension estimates derived from symplectic geometry, but it also follows from results of Ovsienko and Futorny [Ovs03], [FO05]. The remaining ingredient is an explicit description of the $l+1$ K-orbits Q on $\mathcal{B}$ with $l(Q) = n-1-l$, and the closely related study of K-orbits on generalized flag varieties G/P. Our theorem has the following consequence. Let $\mathfrak{b}_+$ denote the Borel subalgebra consisting of upper triangular matrices. For $i = 1, \ldots, n$, let $(i\,n)$ be the permutation matrix corresponding to the transposition interchanging i and n, and let $\mathfrak{b}_i := \mathrm{Ad}(i\,n)\mathfrak{b}_+$.

Corollary 1.2. *If $x \in \mathfrak{g}(l)$, then x is K-conjugate to an element in one of $l+1$ explicitly determined θ-stable parabolic subalgebras. In particular, if $x \in \mathfrak{g}(n-1)$, then x is K-conjugate to an element of $\mathfrak{b}_i$, where $i = 1, \ldots, n$.*

This paper is part of a series of papers on K-orbits on $\mathcal{B}$ and the Gelfand-Zeitlin system. In [CE12], we used K-orbits to determine the so-called strongly regular elements in the nilfiber of the moment map of the Gelfand-Zeitlin system. These are matrices $x \in \mathfrak{g}$ such that x_i is nilpotent for all $i = 1, \ldots, n$ with the added condition that the differentials of the Gelfand-Zeitlin functions are linearly independent at x. The strongly regular elements were first studied extensively in [KW06]. In later work, we will refine Corollary 1.2 to provide a standard form for all elements of $\mathfrak{g}(l)$. This uses K-orbits and a finer study of the algebraic geometry of the varieties $\mathfrak{g}(l)$. In particular, we will give a more conceptual proof of the main result from [Col11] and use K-orbits to describe the geometry of arbitrary fibers of the moment map for the Gelfand-Zeitlin system.

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2. Preliminaries

We show flatness of the partial Kostant-Wallach morphism and recall needed results concerning K-orbits on $\mathcal{B}$.

2.1. **The partial Kostant-Wallach map.** For $x \in \mathfrak{g}$ and $i = 1, \ldots n$, let $x_i \in \mathfrak{gl}(i, \mathbb{C})$ denote the upper left $i \times i$ corner of the matrix x. For any $y \in \mathfrak{gl}(i, \mathbb{C})$, let $tr(y)$ denote the trace of y. For $j = 1, \ldots, i$, let $f_{i,j}(x) = tr((x_i)^j)$, which is a homogeneous function of degree j on $\mathfrak{g}$. The Gelfand-Zeitlin collection of functions is the set $J_{GZ} = \{f_{i,j}(x) : i = 1, \ldots, n, j = 1, \ldots, i\}$. The restriction of these functions to any regular adjoint orbit in $\mathfrak{g}$ produces an integrable system on the orbit [KW06]. Let $\chi_{i,j} : \mathfrak{gl}(i, \mathbb{C}) \to \mathbb{C}$ be the function $\chi_{i,j}(y) = tr(y^j)$, so that $f_{i,j}(x) = \chi_{i,j}(x_i)$ and $\chi_i := (\chi_{i,1}, \ldots, \chi_{i,i})$ is the adjoint quotient for $\mathfrak{gl}(i, \mathbb{C})$. The Kostant-Wallach map is the morphism given by

$$\Phi : \mathfrak{g} \to \mathbb{C}^1 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^n;\ \Phi(x) = (\chi_1(x_1), \ldots, \chi_n(x)). \tag{2.1}$$

We will also consider the partial Kostant-Wallach map given by the morphism

$$\Phi_n : \mathfrak{g} \to \mathbb{C}^{n-1} \times \mathbb{C}^n;\ \Phi_n(x) = (\chi_{n-1}(x_{n-1}), \chi_n(x)). \tag{2.2}$$

Note that

$$\Phi_n = pr \circ \Phi, \tag{2.3}$$

where $pr : \mathbb{C}^1 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^n \to \mathbb{C}^{n-1} \times \mathbb{C}^n$ is projection on the last two factors.

Remark 2.1. *By Theorem 0.1 of* [KW06]*, the map Φ is surjective, and it follows easily that Φ_n is surjective.*

We let $I_n = (\{f_{ij}\}_{i=n-1,n;j=1,\ldots,i})$ denote the ideal generated by the functions $J_{GZ,n} := \{f_{i,j} : i = n-1, n;\ j = 1, \ldots, i\}$. We call the vanishing set $V(I_n)$ the variety of *partially strongly nilpotent matrices* and denote it by SN_n. Thus,

$$SN_n := \{x \in \mathfrak{g} :\ x,\, x_{n-1} \text{ are nilpotent}\}. \tag{2.4}$$

We let $\Gamma_n := \mathbb{C}[\{f_{ij}\}_{i=n-1,n;j=1,\ldots,i}]$ be the subring of regular functions on $\mathfrak{g}$ generated by $J_{GZ,n}$.

Recall that if $Y \subset \mathbb{C}^m$ is a closed equidimensional subvariety of dimension $m-d$, then Y is called a complete intersection if $Y = V(f_1, \ldots, f_d)$ is the vanishing set of d functions.

Theorem 2.2. *The variety of partially strongly nilpotent matrices SN_n is a complete intersection of dimension*

$$d_n := n^2 - 2n + 1. \tag{2.5}$$

Before proving Theorem 2.2, we show how it implies the flatness of the partial Kostant-Wallach map Φ_n.

Proposition 2.3.
(1) *For all $c \in \mathbb{C}^{n-1} \times \mathbb{C}^n$, $\dim(\Phi_n^{-1}(c)) = n^2-2n+1$. Thus, $\Phi_n^{-1}(c)$ is a complete intersection.*
(2) *The partial Kostant-Wallach map $\Phi_n : \mathfrak{g} \to \mathbb{C}^{2n-1}$ is a flat morphism. Thus, $\mathbb{C}[\mathfrak{g}]$ is flat over Γ_n.*

Proof. For $x \in \mathfrak{g}$, we let d_x be the maximum of the dimensions of irreducible components of $\Phi_n^{-1}(\Phi_n(x))$. For $c \in \mathbb{C}^{n-1} \times \mathbb{C}^n$, each irreducible component of $\Phi_n^{-1}(c)$ has dimension at least d_n since $\Phi_n^{-1}(c)$ is defined by $2n-1$ equations in $\mathfrak{g}$. Hence, $d_x \geq d_n$. Since the functions $f_{i,j}$ are homogeneous, it follows that scalar multiplication by $\lambda \in \mathbb{C}^\times$ induces an isomorphism $\Phi_n^{-1}(\Phi_n(x)) \to \Phi_n^{-1}(\Phi_n(\lambda x))$. It follows that $d_x = d_{\lambda x}$. By upper semi-continuity of dimension (see for example, Proposition 4.4 of [Hum75]), the set of $y \in \mathfrak{g}$ such that $d_y \geq d$ is closed for each integer d. It follows that $d_0 \geq d_x$. By Theorem 2.2, $d_0 = d_n$. The first assertion follows easily. The second assertion now follows by the corollary to Theorem 23.1 of [Mat86].

Q.E.D.

Remark 2.4. *We note that Proposition 2.3 implies that $\mathbb{C}[\mathfrak{g}]$ is free over Γ_n. This follows from a result in commutative algebra. Let $A = \oplus_{n\geq 0} A_n$ be a graded ring with $A_0 = k$ a field and let $M = \oplus_{n\geq 0} M_n$ be a graded A-module. The needed result asserts that M is flat over A if and only if M is free over A. One direction of this assertion is obvious, and the*

other direction may be proved using the same argument as in the proof of Proposition 20 on page 73 of [Ser00]*, which is the analogous assertion for finitely generated modules over local rings. In* [Ser00]*, the assumption that M is finitely generated over A is needed only to apply Nakayama's lemma, but in our graded setting, Nakayama's lemma (with ideal $I = \oplus_{n>0} A_n$) does not require the module M to be finitely generated.*

Remark 2.5. *Let $I = (J_{GZ})$ be the ideal in $\mathbb{C}[\mathfrak{g}]$ generated by the Gelfand-Zeitlin collection of functions J_{GZ}, and let $SN = V(I)$ be the strongly nilpotent matrices, i.e., $SN = \{x \in \mathfrak{g} : x_i \text{ is nilpotent for } i = 1, \ldots, n\}$. Ovsienko proves in* [Ovs03] *that SN is a complete intersection, and results of Futorny and Ovsienko from* [FO05] *show that Ovsienko's theorem implies that $\mathbb{C}[\mathfrak{g}]$ is free over $\Gamma := \mathbb{C}[\{f_{i,j}\}_{i=1,\ldots,n;\, j=1,\ldots,i}]$. It then follows easily that $\mathbb{C}[\mathfrak{g}]$ is flat over Γ_n, and hence that Φ_n is flat. Although we could have simply cited the results of Futorny and Ovsienko to prove flatness of Φ_n, we prefer our approach, which we regard as more conceptual.*

Proof of Theorem 2.2. Let $\mathfrak{X}$ be an irreducible component of SN_n. We observed in the proof of Proposition 2.3 that $\dim \mathfrak{X} \geq d_n$. To show $\dim \mathfrak{X} \leq d_n$, we consider a generalization of the Steinberg variety (see Section 3.3 of [CG97]). We first recall a few facts about the cotangent bundle to the flag variety.

For the purposes of this proof, we denote the flag variety of $\mathfrak{gl}(n, \mathbb{C})$ by $\mathcal{B}_n$. We consider the form $<< \cdot, \cdot >>$ on $\mathfrak{g}$ given by $<< x, y >>= tr(xy)$ for $x, y \in \mathfrak{g}$. If $\mathfrak{b} \in \mathcal{B}_n$, the annihilator $\mathfrak{b}^{\perp}$ of $\mathfrak{b}$ with respect to the form $<< \cdot, \cdot >>$ is $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. We can then identify $T^*(\mathcal{B}_n)$ with the closed subset of $\mathfrak{g} \times \mathcal{B}_n$ given by:

$$T^*(\mathcal{B}_n) = \{(x, \mathfrak{b}) : \mathfrak{b} \in \mathcal{B}_n, x \in \mathfrak{n}\}.$$

We let $\mathfrak{g}_{n-1} = \mathfrak{gl}(n-1, \mathbb{C})$ and view $\mathfrak{g}_{n-1}$ as a subalgebra of $\mathfrak{g}$ by embedding $\mathfrak{g}_{n-1}$ in the top lefthand corner of $\mathfrak{g}$. Since $\mathfrak{g}$ is the direct sum $\mathfrak{g} = \mathfrak{g}_{n-1} \oplus \mathfrak{g}_{n-1}^{\perp}$, the restriction of $<< \cdot, \cdot >>$ to $\mathfrak{g}_{n-1}$ is non-degenerate. For a Borel subalgebra $\mathfrak{b}' \in \mathcal{B}_{n-1}$, we let $\mathfrak{n}' = [\mathfrak{b}', \mathfrak{b}']$. We consider a closed subvariety $Z \subset \mathfrak{g} \times \mathcal{B}_n \times \mathcal{B}_{n-1}$ defined as follows:

$$Z = \{(x, \mathfrak{b}, \mathfrak{b}') : \mathfrak{b} \in \mathcal{B}_n \text{ , } \mathfrak{b}' \in \mathcal{B}_{n-1} \text{ and } x \in \mathfrak{n},\ x_{n-1} \in \mathfrak{n}'\}. \tag{2.6}$$

Consider the morphism $\mu : Z \to \mathfrak{g}$, where $\mu(x, \mathfrak{b}, \mathfrak{b}') = x$. Since the varieties $\mathcal{B}_n$ and $\mathcal{B}_{n-1}$ are projective, the morphism μ is proper.

We consider the closed embedding $Z \hookrightarrow T^*(\mathcal{B}_n) \times T^*(\mathcal{B}_{n-1}) \cong T^*(\mathcal{B}_n \times \mathcal{B}_{n-1})$ given by $(x, \mathfrak{b}, \mathfrak{b}') \to (x, -x_{n-1}, \mathfrak{b}, \mathfrak{b}')$. We denote the image of Z under this embedding by $\widetilde{Z} \subset T^*(\mathcal{B}_n \times \mathcal{B}_{n-1})$. Let G_{n-1} be the closed subgroup of $GL(n, \mathbb{C})$ corresponding to $\mathfrak{g}_{n-1}$. Then G_{n-1} acts diagonally on $\mathcal{B}_n \times \mathcal{B}_{n-1}$ via $k \cdot (\mathfrak{b}, \mathfrak{b}') = (k \cdot \mathfrak{b}, k \cdot \mathfrak{b}')$ for $k \in G_{n-1}$. We claim $\widetilde{Z} \subset T^*(\mathcal{B}_n \times \mathcal{B}_{n-1})$ is the union of conormal bundles to the G_{n-1}-diagonal orbits in $\mathcal{B}_n \times \mathcal{B}_{n-1}$. Indeed, let $(\mathfrak{b}, \mathfrak{b}') \in \mathcal{B}_n \times \mathcal{B}_{n-1}$, and let Q be its G_{n-1}-orbit. Then

$$T_{(\mathfrak{b}, \mathfrak{b}')}(Q) = \text{span}\{(Y \bmod \mathfrak{b},\ Y \bmod \mathfrak{b}') : Y \in \mathfrak{g}_{n-1}\}.$$

Now let $(\lambda_1, \lambda_2) \in (\mathfrak{n}, \mathfrak{n}')$ with $(\lambda_1, \lambda_2) \in (T_Q^*)(\mathcal{B}_n \times \mathcal{B}_{n-1})_{(\mathfrak{b}, \mathfrak{b}')}$, the fiber of the conormal bundle to Q in $\mathcal{B}_n \times \mathcal{B}_{n-1}$ at the point $(\mathfrak{b}, \mathfrak{b}')$. Then

$$<< \lambda_1, Y >> + << \lambda_2, Y >>= 0 \text{ for all } Y \in \mathfrak{g}_{n-1}.$$

Thus, $\lambda_1 + \lambda_2 \in \mathfrak{g}_{n-1}^{\perp}$. But since $\lambda_2 \in \mathfrak{n}' \subset \mathfrak{g}_{n-1}$, it follows that $\lambda_2 = -(\lambda_1)_{n-1}$. Thus,

$$T_Q^*(\mathcal{B}_n \times \mathcal{B}_{n-1}) = \{(\mu_1, \mathfrak{b}_1, -(\mu_1)_{n-1}, \mathfrak{b}_2),\ \mu_1 \in \mathfrak{n}_1,\ (\mu_1)_{n-1} \in \mathfrak{n}_2,\ \text{where } (\mathfrak{b}_1, \mathfrak{b}_2) \in Q\}.$$

We recall the well-known fact that there are only finitely many G_{n-1}-diagonal orbits in $\mathcal{B}_n \times \mathcal{B}_{n-1}$, which follows from [VK78], [Bri87], or in a more explicit form is proved in [Has04]. Therefore, the irreducible component decomposition of $\widetilde{Z}$ is:

$$\widetilde{Z} = \bigcup_i \overline{T_{Q_i}^*(\mathcal{B}_n \times \mathcal{B}_{n-1})} \subset T^*(\mathcal{B}_n \times \mathcal{B}_{n-1}),$$

where i runs over the distinct G_{n-1}-diagonal orbits in $\mathcal{B}_n \times \mathcal{B}_{n-1}$. Thus, $Z \cong \widetilde{Z}$ is a closed, equidimensional subvariety of dimension

$$\dim Z = \frac{1}{2}(\dim T^*(\mathcal{B}_n \times \mathcal{B}_{n-1})) = d_n.$$

Note that $\mu : Z \to SN_n$ is surjective. Since μ is proper, for every irreducible component $\mathfrak{X} \subset SN_n$ of SN_n, we see that

$$\mathfrak{X} = \mu(Z_i) \tag{2.7}$$

for some irreducible component $Z_i \subset Z$. Since $\dim Z_i = d_n$ and $\dim \mathfrak{X} \geq d_n$, we conclude that $\dim \mathfrak{X} = d_n$.

Q.E.D.

In Proposition 3.10, we will determine the irreducible components of SN_n explicitly.

2.2. K**-orbits.** We recall some basic facts about K-orbits on generalized flag varieties G/P (see [Mat79, RS90, MŌ90, Yam97, CE] for more details).

By the general theory of orbits of symmetric subgroups on generalized flag varieties, K has finitely many orbits on $\mathcal{B}$. For this paper, it is useful to parametrize the orbits. To do this, we let B_+ be the upper triangular Borel subgroup of G, and identify $\mathcal{B} \cong G/B_+$ with the variety of flags in $\mathbb{C}^n$. We use the following notation for flags in $\mathbb{C}^n$. Let

$$\mathcal{F} = (V_0 = \{0\} \subset V_1 \subset \cdots \subset V_i \subset \cdots \subset V_n = \mathbb{C}^n).$$

be a flag in $\mathbb{C}^n$, with $\dim V_i = i$ and $V_i = \text{span}\{v_1, \ldots, v_i\}$, with each $v_j \in \mathbb{C}^n$. We will denote the flag $\mathcal{F}$ as follows:

$$v_1 \subset v_2 \subset \cdots \subset v_i \subset v_{i+1} \subset \cdots \subset v_n.$$

We denote the standard ordered basis of $\mathbb{C}^n$ by $\{e_1, \ldots, e_n\}$, and let $E_{i,j} \in \mathfrak{g}$ be the matrix with 1 in the (i, j)-entry and 0 elsewhere.

There are n closed K-orbits on $\mathcal{B}$ (see Example 4.16 of [CE]), $Q_{i,i} = K \cdot \mathfrak{b}_{i,i}$ for $i = 1, \dots, n$, where the Borel subalgebra $\mathfrak{b}_{i,i}$ is the stabilizer of the following flag in $\mathbb{C}^n$:

$$\mathcal{F}_{i,i} = (e_1 \subset \cdots \subset e_{i-1} \subset \underbrace{e_n}_{i} \subset e_i \subset \cdots \subset e_{n-1}). \tag{2.8}$$

Note that if $i = n$, then the flag $\mathcal{F}_{i,i}$ is the standard flag $\mathcal{F}_+$:

$$\mathcal{F}_+ = (e_1 \subset \cdots \subset e_n), \tag{2.9}$$

and $\mathfrak{b}_{n,n} = \mathfrak{b}_+$ is the standard Borel subalgebra of $n \times n$ upper triangular matrices. It is not difficult to check that $K \cdot \mathfrak{b}_{i,i} = K \cdot \mathrm{Ad}(i\, n)\mathfrak{b}_+$. If $i = 1$, then $K \cdot \mathfrak{b}_{1,1} = K \cdot \mathfrak{b}_-$, where $\mathfrak{b}_-$ is the Borel subalgebra of lower triangular matrices in $\mathfrak{g}$.

The non-closed K-orbits in $\mathcal{B}$ are the orbits $Q_{i,j} = K \cdot \mathfrak{b}_{i,j}$ for $1 \leq i < j \leq n$, where $\mathfrak{b}_{i,j}$ is the stabilizer of the flag in $\mathbb{C}^n$:

$$\mathcal{F}_{i,j} = (e_1 \subset \cdots \subset \underbrace{e_i + e_n}_{i} \subset e_{i+1} \subset \cdots \subset e_{j-1} \subset \underbrace{e_i}_{j} \subset e_j \subset \cdots \subset e_{n-1}). \tag{2.10}$$

There are $\binom{n}{2}$ such orbits (see Notation 4.23 and Example 4.31 of [CE]).

Let w and σ be the permutation matrices corresponding respectively to the cycles $(n\, n-1 \cdots i)$ and $(i+1\, i+2 \dots j)$, and let u_{α_i} be the Cayley transform matrix such that

$$u_{\alpha_i}(e_i) = e_i + e_{i+1},\ u_{\alpha_i}(e_{i+1}) = -e_i + e_{i+1},\ u_{\alpha_i}(e_k) = e_k, k \neq i, i+1.$$

For $1 \leq i \leq j \leq n$, we define:

$$v_{i,j} := \left\{ \begin{array}{ccc} w & \text{if} & i = j \\ w u_{\alpha_i} \sigma & \text{if} & i \neq j \end{array} \right\} \tag{2.11}$$

It is easy to verify that $v_{i,j}(\mathcal{F}_+) = \mathcal{F}_{i,j}$, and thus $\mathrm{Ad}(v_{i,j})\mathfrak{b}_+ = \mathfrak{b}_{i,j}$ (see Example 4.30 of [CE]).

Remark 2.6. *The length of the K-orbit $Q_{i,j}$ is $l(Q_{i,j}) = j - i$ for any $1 \leq i \leq j \leq n$ (see Example 4.30 of* [CE]*). For example, a K-orbit $Q_{i,j}$ is closed if and only if $Q = Q_{i,i}$ for some i. The $n - l$ orbits of length l are $Q_{i,i+l}$, $i = 1, \dots, n-l$.*

For a parabolic subgroup P of G with Lie algebra $\mathfrak{p}$, we consider the generalized flag variety G/P, which we identify with parabolic subalgebras of type $\mathfrak{p}$ and with partial flags of type $\mathfrak{p}$. We will make use of the following notation for partial flags. Let

$$\mathcal{P} = (V_0 = \{0\} \subset V_1 \subset \cdots \subset V_i \subset \cdots \subset V_k = \mathbb{C}^n)$$

denote a k-step partial flag with $\dim V_j = i_j$ and $V_j = \mathrm{span}\{v_1, \dots, v_{i_j}\}$ for $j = 1, \dots, k$. Then we denote $\mathcal{P}$ as

$$v_1, \dots, v_{i_1} \subset v_{i_1+1}, \dots, v_{i_2} \subset \cdots \subset v_{i_{k-1}+1}, \dots, v_{i_k}.$$

In particular for $i \leq j$, we let $\mathfrak{r}_{i,j} \subset \mathfrak{g}$ denote the parabolic subalgebra which is the stabilizer of the $n - (j-i)$-step partial flag in $\mathbb{C}^n$

$$\mathcal{R}_{i,j} = (e_1 \subset e_2 \subset \cdots \subset e_{i-1} \subset e_i, \dots, e_j \subset e_{j+1} \subset \cdots \subset e_n). \tag{2.12}$$

It is easy to see that $\mathfrak{r}_{i,j}$ is the standard parabolic subalgebra generated by the Borel subalgebra $\mathfrak{b}_+$ and the negative simple root spaces $\mathfrak{g}_{-\alpha_i}, \mathfrak{g}_{-\alpha_{i+1}}, \dots, \mathfrak{g}_{-\alpha_{j-1}}$. We note that $\mathfrak{r}_{i,j}$ has Levi decomposition $\mathfrak{r}_{i,j} = \mathfrak{m} + \mathfrak{n}$, with $\mathfrak{m}$ consisting of block diagonal matrices of the form

$$\mathfrak{m} = \underbrace{\mathfrak{gl}(1,\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(1,\mathbb{C})}_{i-1 \text{ factors}} \oplus \mathfrak{gl}(j+1-i,\mathbb{C}) \oplus \underbrace{\mathfrak{gl}(1,\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(1,\mathbb{C})}_{n-j \text{ factors}}. \tag{2.13}$$

Let $R_{i,j}$ be the parabolic subgroup of G with Lie algebra $\mathfrak{r}_{i,j}$. Let $\mathfrak{p}_{i,j} := \mathrm{Ad}(v_{i,j})\mathfrak{r}_{i,j} \in G/R_{i,j}$, where $v_{i,j}$ is defined in (2.11). Then $\mathfrak{p}_{i,j}$ is the stabilizer of the partial flag

$$\mathcal{P}_{i,j} = (e_1 \subset e_2 \subset \cdots \subset e_{i-1} \subset e_i, \dots, e_{j-1}, e_n \subset e_j \subset \cdots \subset e_{n-1}), \tag{2.14}$$

and $\mathfrak{p}_{i,j} \in G/R_{i,j}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$. Indeed, recall that θ is given by conjugation by the diagonal matrix $d = diag[1, \dots, 1, -1]$. Clearly $d(\mathcal{P}_{i,j}) = \mathcal{P}_{i,j}$, whence $\mathfrak{p}_{i,j}$ is θ-stable. Moreover, the parabolic subalgebra $\mathfrak{p}_{i,j}$ has Levi decomposition $\mathfrak{p}_{i,j} = \mathfrak{l} \oplus \mathfrak{u}$ where both $\mathfrak{l}$ and $\mathfrak{u}$ are θ-stable and $\mathfrak{l}$ is isomorphic to the Levi subalgebra in Equation (2.13). Since $\mathfrak{p}_{i,j}$ is θ-stable, it follows from Theorem 2 of [BH00] that the K-orbit $Q_{\mathfrak{p}_{i,j}} = K \cdot \mathfrak{p}_{i,j}$ is closed in $G/R_{i,j}$.

For a parabolic subgroup $P \subset G$ with Lie algebra $\mathfrak{p} \subset \mathfrak{g}$, consider the partial Grothendieck resolution $\tilde{\mathfrak{g}}^{\mathfrak{p}} = \{(x, \mathfrak{r}) \in \mathfrak{g} \times G/P \mid x \in \mathfrak{r}\}$, as well as the morphisms $\mu : \tilde{\mathfrak{g}}^{\mathfrak{p}} \to \mathfrak{g}$, $\mu(x, \mathfrak{r}) = x$, and $\pi : \tilde{\mathfrak{g}}^{\mathfrak{p}} \to G/P$, $\pi(x, \mathfrak{r}) = \mathfrak{r}$. Then π is a smooth morphism of relative dimension $\dim \mathfrak{p}$ (for G/B, see Section 3.1 of [CG97] and Proposition III.10.4 of [Har77], and the general case of G/P follows by the same argument). For $\mathfrak{r} \in G/P$, let $Q_{\mathfrak{r}} = K \cdot \mathfrak{r} \subset G/P$. Then $\pi^{-1}(Q_{\mathfrak{r}})$ has dimension $\dim(Q_{\mathfrak{r}}) + \dim(\mathfrak{r})$. It is well-known that μ is proper and its restriction to $\pi^{-1}(Q_{\mathfrak{r}})$ generically has finite fibers (Proposition 3.1.34 and Example 3.1.35 of [CG97] for the case of G/B, and again the general case has a similar proof).

Notation 2.7. *For a parabolic subalgebra $\mathfrak{r}$ with K-orbit $Q_{\mathfrak{r}} \in G/P$, we consider the irreducible subset*

$$Y_{\mathfrak{r}} := \mu(\pi^{-1}(Q_{\mathfrak{r}})) = \mathrm{Ad}(K)\mathfrak{r}. \tag{2.15}$$

To emphasize the orbit $Q_{\mathfrak{r}}$, we will also denote this set as

$$Y_{Q_{\mathfrak{r}}} := Y_{\mathfrak{r}}. \tag{2.16}$$

It follows from generic finiteness of μ that $Y_{Q_{\mathfrak{r}}}$ contains an open subset of dimension

$$\dim(Y_{Q_{\mathfrak{r}}}) := \dim \pi^{-1}(Q_{\mathfrak{r}}) = \dim \mathfrak{r} + \dim(Q_{\mathfrak{r}}) = \dim \mathfrak{r} + \dim(\mathfrak{k}/\mathfrak{k} \cap \mathfrak{r}), \tag{2.17}$$

where $\mathfrak{k} = Lie(K) = \mathfrak{gl}(n-1,\mathbb{C}) \oplus \mathfrak{gl}(1,\mathbb{C})$.

Remark 2.8. *Since μ is proper, when $Q_{\mathfrak{r}} = K \cdot \mathfrak{r}$ is closed in G/P, then $Y_{Q_{\mathfrak{r}}}$ is closed.*

Remark 2.9. *Note that*

$$\mathfrak{g} = \bigcup_{Q \subset G/P} Y_Q,$$

is a partition of $\mathfrak{g}$, where the union is taken over the finitely many K-orbits in G/P.

Lemma 2.10. *Let $Q \subset G/P$ be a K-orbit. Then*

$$\overline{Y_Q} = \bigcup_{Q' \subset \overline{Q}} Y_{Q'}. \tag{2.18}$$

Proof. Since π is a smooth morphism, it is flat by Theorem III.10.2 of [Har77]. Thus, by Theorem VIII.4.1 of [Gro03], $\pi^{-1}(\overline{Q}) = \overline{\pi^{-1}(Q)}$. The result follows since μ is proper.

Q.E.D.

2.3. **Comparison of $K \cdot \mathfrak{b}_{i,j}$ and $K \cdot \mathfrak{p}_{i,j}$.** We prove a technical result that will be needed to prove our main theorem.

Remark 2.11. *Note that $\mathfrak{b}_{i,j} \subset \mathfrak{p}_{i,j}$ and when $i = j$, $\mathfrak{p}_{i,i}$ is the Borel subalgebra $\mathfrak{b}_{i,i}$. To check the first assertion, note that $\mathfrak{b}_+ \subset \mathfrak{r}_{i,j}$ so that $\mathfrak{b}_{i,j} = \mathrm{Ad}(v_{i,j})\mathfrak{b}_+ \subset \mathrm{Ad}(v_{i,j})\mathfrak{r}_{i,j} = \mathfrak{p}_{i,j}$. The second assertion is verified by noting that when $i = j$, the partial flag $\mathcal{P}_{i,j}$ is the full flag $\mathcal{F}_{i,i}$.*

Proposition 2.12. *Consider the K-orbits $Q_{i,j} = K \cdot \mathfrak{b}_{i,j} \subset \mathcal{B}$ and $Q_{\mathfrak{p}_{i,j}} = K \cdot \mathfrak{p}_{i,j} \subset G/P_{i,j}$, with $1 \leq i \leq j \leq n$. Then $\dim(Y_{\mathfrak{b}_{i,j}}) = \dim(Y_{\mathfrak{p}_{i,j}})$ and $\overline{Y_{\mathfrak{b}_{i,j}}} = Y_{\mathfrak{p}_{i,j}}$.*

Proof. By definitions and Remark 2.11, $Y_{\mathfrak{b}_{i,j}}$ is a constructible subset of $Y_{\mathfrak{p}_{i,j}}$. Since $Y_{\mathfrak{p}_{i,j}}$ is closed by Remark 2.8, and irreducible by construction, it suffices to show that $\dim(Y_{\mathfrak{b}_{i,j}}) = \dim(Y_{\mathfrak{p}_{i,j}})$.

We compute the dimension of $Y_{\mathfrak{b}_{i,j}}$ using Equation (2.17). Since $l(Q_{i,j}) = j - i$, it follows that $\dim Q_{i,j} = \dim \mathcal{B}_{n-1} + j - i$. Since $\dim(\mathcal{B}_{n-1}) = \binom{n-1}{2}$, Equation (2.17) then implies:

$$\begin{aligned} \dim Y_{\mathfrak{b}_{i,j}} = \dim \mathfrak{b}_{i,j} + \dim \mathcal{B}_{n-1} + l(Q_{i,j}) &= \binom{n+1}{2} + \binom{n-1}{2} + l(Q_{i,j}) \\ &= n^2 - n + 1 + j - i. \end{aligned} \tag{2.19}$$

We now compute the dimension of $Y_{\mathfrak{p}_{i,j}}$. By Equation (2.17), it follows that

$$\dim Y_{\mathfrak{p}_{i,j}} = \dim \mathfrak{p}_{i,j} + \dim \mathfrak{k} - \dim(\mathfrak{k} \cap \mathfrak{p}_{i,j}). \tag{2.20}$$

Since both $\mathfrak{l}$ and $\mathfrak{u}$ are θ-stable, it follows that $\dim \mathfrak{k} \cap \mathfrak{p}_{i,j} = \dim \mathfrak{k} \cap \mathfrak{l} + \dim \mathfrak{k} \cap \mathfrak{u}$. To compute these dimensions, it is convenient to use the following explicit matrix description of the parabolic subalgebra $\mathfrak{p}_{i,j}$, which follows from Equation (2.14).

(2.21)

$$\mathfrak{p}_{i,j} = \begin{bmatrix} a_{11} & \dots & \dots & a_{1i-1} & a_{1i} & \dots & a_{1j-1} & \dots & \dots & a_{1n-1} & a_{1n} \\ 0 & \ddots & & \vdots & \vdots & * & \vdots & * & * & \vdots & \vdots \\ \vdots & & & a_{i-1i-1} & \vdots & * & \vdots & * & * & a_{i-1n-1} & a_{i-1n} \\ & & & 0 & a_{ii} & \dots & a_{ij-1} & * & * & a_{in-1} & a_{in} \\ & & & \vdots & \vdots & \ddots & \vdots & * & * & \vdots & \vdots \\ & & & \vdots & a_{ij-1} & \dots & a_{j-1j-1} & \dots & \dots & a_{j-1n-1} & a_{j-1n} \\ & & & 0 & 0 & \dots & 0 & a_{jj} & \dots & a_{jn-1} & 0 \\ & & & \vdots & \vdots & & \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & & & \vdots & 0 & & 0 & 0 & 0 & a_{n-1n-1} & 0 \\ 0 & \dots & \dots & 0 & a_{ni} & \dots & a_{nj-1} & a_{nj} & \dots & a_{nn-1} & a_{nn} \end{bmatrix}.$$

Using (2.21), we observe that $\mathfrak{k} \cap \mathfrak{l} \cong \mathfrak{gl}(1,\mathbb{C})^{n-j+i} \oplus \mathfrak{gl}(j-i,\mathbb{C})$, so that $\dim \mathfrak{k} \cap \mathfrak{l} = n-j+i+(j-i)^2$. Now $\mathfrak{u} = \mathfrak{u} \cap \mathfrak{k} \oplus \mathfrak{u}^{-\theta}$, where $\mathfrak{u}^{-\theta} := \{x \in \mathfrak{u} : \theta(x) = -x\}$. Using (2.21), we see that $\mathfrak{u}^{-\theta}$ has basis $\{E_{n,j}, \dots, E_{n,n-1}, E_{1,n}, \dots, E_{i-1,n}\}$, so $\dim \mathfrak{u}^{-\theta} = n-j+i-1$. Thus, $\dim \mathfrak{u} \cap \mathfrak{k} = \dim \mathfrak{u} - (n-j+i-1)$. Putting these observations together, we obtain

$$\dim \mathfrak{k} \cap \mathfrak{p}_{i,j} = (j-i)^2 + \dim \mathfrak{u} + 1. \tag{2.22}$$

Now

$$\dim \mathfrak{p}_{i,j} = \dim \mathfrak{l} + \dim \mathfrak{u} = (j-i+1)^2 + n - j + i - 1 + \dim \mathfrak{u}.$$

(see Equation (2.13)). Thus, Equation (2.20) implies that

$$\dim Y_{\mathfrak{p}_{i,j}} = \dim \mathfrak{k} + (j-i+1)^2 + n - j + i - 1 - (j-i)^2 - 1 = n^2 - n + 1 + j - i,$$

which agrees with (2.19), and hence completes the proof.

Q.E.D.

Remark 2.13. *It follows from Equation (2.21) that* $(\mathfrak{p}_{i,j})_{n-1} := \pi_{n-1,n}(\mathfrak{p}_{i,j})$ *is a parabolic subalgebra, where* $\pi_{n,n-1} : \mathfrak{g} \to \mathfrak{gl}(n-1,\mathbb{C})$ *is the projection* $x \mapsto x_{n-1}$. *Further, with* $l = j-i$, $(\mathfrak{p}_{i,j})_{n-1}$ *has Levi decomposition* $(\mathfrak{p}_{i,j})_{n-1} = \mathfrak{l}_{n-1} \oplus \mathfrak{u}_{n-1}$ *with* $\mathfrak{l}_{n-1} = \mathfrak{gl}(1,\mathbb{C})^{n-1-l} \oplus \mathfrak{gl}(l,\mathbb{C})$.

3. The varieties $\mathfrak{g}(l)$

In this section, we prove our main results.

For $x \in \mathfrak{g}$, let $\sigma(x) = \{\lambda_1, \dots, \lambda_n\}$ denote its eigenvalues, where an eigenvalue λ is listed k times if it appears with multiplicity k. Similarly, let $\sigma(x_{n-1}) = \{\mu_1, \dots, \mu_{n-1}\}$ be the eigenvalues of $x_{n-1} \in \mathfrak{gl}(n-1,\mathbb{C})$, again listed with multiplicity. For $i = n-1, n$, let $\mathfrak{h}_i \subset \mathfrak{g}_i := \mathfrak{gl}(i,\mathbb{C})$ be the standard Cartan subalgebra of diagonal matrices. We denote

elements of $\mathfrak{h}_{n-1}\times\mathfrak{h}_n$ by (x,y), with $x=(x_1,\dots,x_{n-1})\in\mathbb{C}^{n-1}$ and $y=(y_1,\dots,y_n)\in\mathbb{C}^n$ the diagonal coordinates of x and y. For $l=0,\dots,n-1$, we define

$$\begin{aligned}(\mathfrak{h}_{n-1}\times\mathfrak{h}_n)(\geq l)=&\{(x,y):\exists\, 1\leq i_1<\cdots<i_l\leq n-1 \text{ with } x_{i_j}=y_{k_j}\\ &\text{for some } 1\leq k_1,\dots,k_l\leq n \text{ with } k_j\neq k_m\}.\end{aligned}$$

Thus, $(\mathfrak{h}_{n-1}\times\mathfrak{h}_n)(\geq l)$ consists of elements of $\mathfrak{h}_{n-1}\times\mathfrak{h}_n$ with at least l coincidences in the spectrum of x and y counting repetitions. Note that $(\mathfrak{h}_{n-1}\times\mathfrak{h}_n)(\geq l)$ is a closed subvariety of $\mathfrak{h}_{n-1}\times\mathfrak{h}_n$ and is equidimensional of codimension l.

Let $W_i=W_i(\mathfrak{g}_i,\mathfrak{h}_i)$ be the Weyl group of $\mathfrak{g}_i$. Then $W_{n-1}\times W_n$ acts on $(\mathfrak{h}_{n-1}\times\mathfrak{h}_n)(\geq l)$. Consider the finite morphism $p:\mathfrak{h}_{n-1}\times\mathfrak{h}_n\to(\mathfrak{h}_{n-1}\times\mathfrak{h}_n)/(W_{n-1}\times W_n)$. Let $F_i:\mathfrak{h}_i/W_i\to\mathbb{C}^i$ be the Chevalley isomorphism, and let

$$V^{n-1,n}:=\mathbb{C}^{n-1}\times\mathbb{C}^n,\ \text{so that } F_{n-1}\times F_n:(\mathfrak{h}_{n-1}\times\mathfrak{h}_n)/(W_{n-1}\times W_n)\to V^{n-1,n}$$

is an isomorphism. The following varieties play a major role in our study of eigenvalue coincidences.

Definition-Notation 3.1. *For $l=0,\dots,n-1$, we let*

$$V^{n-1,n}(\geq l):=(F_{n-1}\times F_n)((\mathfrak{h}_{n-1}\times\mathfrak{h}_n)(\geq l)/(W_{n-1}\times W_n)), \tag{3.1}$$

$$V^{n-1,n}(l):=V^{n-1,n}(\geq l)\setminus V^{n-1,n}(\geq l+1). \tag{3.2}$$

For convenience, we let $V^{n-1,n}(n)=\emptyset$.

Lemma 3.2. *The set $V^{n-1,n}(\geq l)$ is an irreducible closed subvariety of $V^{n-1,n}$ of dimension $2n-1-l$. Further, $V^{n-1,n}(l)$ is open and dense in $V^{n-1,n}(\geq l)$.*

Proof. Indeed, the set $Y:=\{(x,y)\in\mathfrak{h}_{n-1}\times\mathfrak{h}_n : x_i=y_i \text{ for } i=1,\dots,l\}$ is closed and irreducible of dimension $2n-1-l$. The first assertion follows since $(F_{n-1}\times F_n)\circ p$ is a finite morphism and $(F_{n-1}\times F_n)\circ p(Y)=V^{n-1,n}(\geq l)$. The last assertion of the lemma now follows from Equation (3.2).

Q.E.D.

Definition 3.3. *We let*

$$\mathfrak{g}(\geq l):=\Phi_n^{-1}(V^{n-1,n}(\geq l)).$$

Remark 3.4. *Recall that the quotient morphism $p_i:\mathfrak{g}_i\to\mathfrak{g}_i//GL(i,\mathbb{C})\cong\mathfrak{h}_i/W_i$ associates to $y\in\mathfrak{g}_i$ its spectrum $\sigma(y)$, and $(F_{n-1}\times F_n)\circ(p_{n-1}\times p_n)=\Phi_n$. It follows that $\mathfrak{g}(\geq l)$ consists of elements of x with at least l coincidences in the spectrum of x and x_{n-1}, counted with multiplicity.*

It is routine to check that

$$\mathfrak{g}(l):=\mathfrak{g}(\geq l)\setminus\mathfrak{g}(\geq l+1)=\Phi_n^{-1}(V^{n-1,n}(l)) \tag{3.3}$$

consists of elements of $\mathfrak{g}$ with exactly l coincidences in the spectrum of x and x_{n-1}, counted with multiplicity.

Proposition 3.5. (1) *The variety $\mathfrak{g}(\geq l)$ is equidimensional of dimension $n^2 - l$.*
(2) $\mathfrak{g}(\geq l) = \overline{\mathfrak{g}(l)} = \bigcup_{k\geq l} \mathfrak{g}(k)$.

Proof. By Proposition 2.3, the morphism Φ_n is flat. By Proposition III.9.5 and Corollary III.9.6 of [Har77], the variety $\mathfrak{g}(\geq l)$ is equidimensional of dimension $\dim(V^{n-1,n}(\geq l)) + (n-1)^2$, which gives the first assertion by Lemma 3.2. For the second assertion, by the flatness of Φ_n, Theorem VIII.4.1 of [Gro03], and Lemma 3.2,

$$\overline{\mathfrak{g}(l)} = \overline{\Phi_n^{-1}(V^{n-1,n}(l))} = \Phi_n^{-1}(\overline{V^{n-1,n}(l)}) = \Phi_n^{-1}(V^{n-1,n}(\geq l)) = \mathfrak{g}(\geq l). \tag{3.4}$$

The remaining equality follows from definitions.

Q.E.D.

We now relate the partitions $\mathfrak{g} = \bigcup \mathfrak{g}(l)$ and $\mathfrak{g} = \bigcup_{Q\subset\mathcal{B}} Y_Q$ (see Remark 2.9).

Theorem 3.6. (1) *Consider the closed subvarieties $Y_{\mathfrak{p}_{i,j}}$ for $1 \leq i \leq j \leq n$, and let $l = j - i$. Then $Y_{\mathfrak{p}_{i,j}} \subset \mathfrak{g}(\geq n-1-l)$.*
(2) *In particular, if $Q \subset \mathcal{B}$ is a K-orbit with $l(Q) = l$, then $Y_Q \subset \mathfrak{g}(\geq n-1-l)$.*

Proof. The second statement of the theorem follows from the first statement using Remark 2.6 and Proposition 2.12.

To prove the first statement of the theorem, let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ with $\mathfrak{q} \in Q_{\mathfrak{p}_{i,j}}$, and let $y \in \mathfrak{q}$. We need to show that $\Phi_n(y) \in V^{n-1,n}(\geq n-1-l)$. Since the map Φ_n is K-invariant, it is enough to show that $\Phi_n(x) \in V^{n-1,n}(\geq n-1-l)$ for $x \in \mathfrak{p}_{i,j}$.

We recall that $\Phi_n(x) = (\chi_{n-1}(x_{n-1}), \chi_n(x))$ where $\chi_i : \mathfrak{gl}(i,\mathbb{C}) \to \mathbb{C}^i$ is the adjoint quotient for $i = n-1, n$. For $x \in \mathfrak{p}_{i,j}$, let $x_{\mathfrak{l}}$ be the projection of x onto $\mathfrak{l}$ off of $\mathfrak{u}$. It is well-known that $\chi_n(x) = \chi_n(x_{\mathfrak{l}})$. Using the identification $\mathfrak{l} \cong \mathfrak{gl}(1,\mathbb{C})^{n-1-l} \oplus \mathfrak{gl}(l+1,\mathbb{C})$, we decompose $x_{\mathfrak{l}}$ as $x_{\mathfrak{l}} = x_{\mathfrak{gl}(1)^{n-1-l}} + x_{\mathfrak{gl}(l+1)}$, where $x_{\mathfrak{gl}(1)^{n-1-l}} \in \mathfrak{gl}(1,\mathbb{C})^{n-1-l}$ and $x_{\mathfrak{gl}(l+1)} \in \mathfrak{gl}(l+1,\mathbb{C})$. It follows that the coordinates of $x_{\mathfrak{gl}(1)^{n-1-l}}$ are in the spectrum of x (see (2.21)).

Recall the projection $\pi_{n,n-1} : \mathfrak{g} \to \mathfrak{g}_{n-1}$, $\pi_{n,n-1}(x) = x_{n-1}$. Recall the Levi decomposition $(\mathfrak{p}_{i,j})_{n-1} = \mathfrak{l}_{n-1} \oplus \mathfrak{u}_{n-1}$ of the parabolic subalgebra $(\mathfrak{p}_{i,j})_{n-1}$ of $\mathfrak{gl}(n-1,\mathbb{C})$ from Remark 2.13, and recall that $\mathfrak{l}_{n-1} = \mathfrak{gl}(1,\mathbb{C})^{n-1-l} \oplus \mathfrak{gl}(l,\mathbb{C})$. Thus, $\chi_{n-1}(x_{n-1}) = \chi_{n-1}((x_{n-1})_{\mathfrak{l}_{n-1}})$. We use the decomposition $(x_{n-1})_{\mathfrak{l}_{n-1}} = x_{\mathfrak{gl}(1)^{n-1-l}} + \pi_{l+1,l}(x_{\mathfrak{gl}(l+1)})$, where $\pi_{l+1,l} : \mathfrak{gl}(l+1,\mathbb{C}) \to \mathfrak{gl}(l,\mathbb{C})$ is the usual projection. It now follows easily from Remark 3.4 that $\Phi_n(x) \in V^{n-1,n}(\geq n-1-l)$, since the coordinates of $x_{\mathfrak{gl}(1)^{n-1-l}}$ are eigenvalues both for x and x_{n-1}.

Q.E.D.

We now recall and prove our main theorem.

Theorem 3.7. *Consider the locally closed subvariety $\mathfrak{g}(n-1-l)$ for $l = 0, \dots, n-1$. Then the decomposition*

$$\mathfrak{g}(n-1-l) = \bigcup_{l(Q)=l} Y_Q \cap \mathfrak{g}(n-1-l), \tag{3.5}$$

is the irreducible component decomposition of the variety $\mathfrak{g}(n-1-l)$, where the union is taken over all K-orbits Q of length l in $\mathcal{B}$. (cf. Theorem (1.1)).

In fact, for $1 \le i \le j \le n$ with $j - i = l$, we have

$$Y_{\mathfrak{b}_{i,j}} \cap \mathfrak{g}(n-1-l) = Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l),$$

so that

$$\mathfrak{g}(n-1-l) = \bigcup_{j-i=l} Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l). \tag{3.6}$$

Proof. We first claim that if $l(Q) = l$, then $Y_Q \cap \mathfrak{g}(n-1-l)$ is non-empty. By Theorem 3.6, $Y_Q \subset \mathfrak{g}(\ge n-1-l)$. Thus, if $Y_Q \cap \mathfrak{g}(n-1-l)$ were empty, then $Y_Q \subset \mathfrak{g}(\ge n-l)$. Hence, by part (1) of Proposition 3.5, $\dim(Y_Q) \le n^2 - n + l$. By Equation (2.19), $\dim(Y_Q) = n^2 - n + l + 1$. This contradiction verifies the claim.

It follows from Equation (3.3) that $\mathfrak{g}(n-1-l)$ is open in $\mathfrak{g}(\ge n-1-l)$. Thus, $Y_Q \cap \mathfrak{g}(n-1-l)$ is a non-empty Zariski open subset of Y_Q, which is irreducible since Y_Q is irreducible.

Now we claim that

$$Y_Q \cap \mathfrak{g}(n-1-l) = \overline{Y_Q} \cap \mathfrak{g}(n-1-l), \tag{3.7}$$

so that $Y_Q \cap \mathfrak{g}(n-1-l)$ is closed in $\mathfrak{g}(n-1-l)$. By Lemma 2.10, $\overline{Y_Q} = \bigcup_{Q' \subset \overline{Q}} Y_{Q'}$. Hence, if (3.7) were not an equality, there would be Q' with $l(Q') < l(Q)$ and $Y_{Q'} \cap \mathfrak{g}(n-1-l)$ nonempty. This contradicts Theorem 3.6, which asserts that $Y_{Q'} \subset \mathfrak{g}(\ge n-l)$, and hence verifies the claim. It follows that $Y_Q \cap \mathfrak{g}(n-1-l)$ is an irreducible, closed subvariety of $\mathfrak{g}(n-1-l)$ of dimension $\dim Y_Q = \dim \mathfrak{g}(n-1-l)$. Thus, $Y_Q \cap \mathfrak{g}(n-1-l)$ is an irreducible component of $\mathfrak{g}(n-1-l)$.

Since $l(Q) = l$, Remark 2.6 implies that $Q = Q_{i,j}$ for some $i \le j$ with $j - i = l$. Then by Proposition 2.12 and Equation (3.7),

$$Y_{\mathfrak{b}_{i,j}} \cap \mathfrak{g}(n-1-l) = Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l). \tag{3.8}$$

Let Z be an irreducible component of $\mathfrak{g}(n-1-l)$. The proof will be complete once we show that $Z = Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l)$ for some i, j with $j - i = l$. To do this, consider the nonempty open set

$$U := \{x \in \mathfrak{g} : \ x_{n-1} \text{ is regular semisimple}\}.$$

Let $\widetilde{U}(n-1-l) := \mathfrak{g}(n-1-l) \cap U$.

Since $\Phi_n : \mathfrak{g} \to V^{n-1,n}$ is surjective (by Remark 2.1), it follows that $\widetilde{U}(n-1-l)$ is a nonempty Zariski open set of $\mathfrak{g}(n-1-l)$. By part (2) of Proposition 2.3 and Exercise III.9.1 of [Har77], $\Phi_n(U) \subset V^{n-1,n}$ is open. Thus, $V^{n-1,n}(n-1-l) \setminus \Phi_n(U)$ is a proper, closed subvariety of $V^{n-1,n}(n-1-l)$ and therefore has positive codimension by Lemma 3.2. It follows by part (2) of Proposition 2.3 and Corollary III.9.6 of [Har77] that $\mathfrak{g}(n-1-l) \setminus \widetilde{U}(n-1-l) = \Phi_n^{-1}(V^{n-1,n}(n-1-l) \setminus \Phi_n(U))$ is a proper, closed subvariety of $\mathfrak{g}(n-1-l)$ of positive codimension. Since $\mathfrak{g}(n-1-l)$ is equidimensional, it follows that $Z \cap \widetilde{U}(n-1-l)$ is nonempty. Thus, it suffices to show that

$$\widetilde{U}(n-1-l) \subset \bigcup_{j-i=l} Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l). \tag{3.9}$$

To prove Equation (3.9), we consider the following subvariety of $\widetilde{U}(n-1-l)$:
(3.10)

$$\Xi = \{x \in \tilde{U}(n-1-l) : \; x_{n-1} = \operatorname{diag}[h_1, \dots, h_{n-1}], \text{ and } \sigma(x_{n-1}) \cap \sigma(x) = \{h_1, \dots, h_{n-1-l}\}\}$$

It is easy to check that any element of $\widetilde{U}(n-1-l)$ is K-conjugate to an element in Ξ. By a linear algebra calculation from Proposition 5.9 of [Col11], elements of Ξ are matrices of the form

$$\begin{bmatrix} h_1 & 0 & \cdots & 0 & y_1 \\ 0 & h_2 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & h_{n-1} & y_{n-1} \\ z_1 & \cdots & \cdots & z_{n-1} & w \end{bmatrix}, \tag{3.11}$$

with $h_i \neq h_j$ for $i \neq j$ and satisfying the equations:

$$\begin{aligned} &z_i y_i = 0 \text{ for } 1 \leq i \leq n-1-l \\ &z_i y_i \in \mathbb{C}^{\times} \text{ for } n-l \leq i \leq n-1. \end{aligned} \tag{3.12}$$

Since the varieties $Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l)$ are K-stable, it suffices to prove

$$\Xi \subset \bigcup_{j-i=l} Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l). \tag{3.13}$$

To prove (3.13), we need to understand the irreducible components of Ξ. For $i = 1, \dots, n-1-l$, we define an index j_i which takes on two values $j_i = U$ (U for upper) or $j_i = L$ (L for lower). Consider the subvariety $\Xi_{j_1,\dots,j_{n-1-l}} \subset \Xi$ defined by:

$$\Xi_{j_1,\dots,j_{n-1-l}} := \{x \in \Xi : z_i = 0 \text{ if } j_i = U, y_i = 0 \text{ if } j_i = L\}. \tag{3.14}$$

Then

$$\Xi = \bigcup_{j_i=U,\,L} \Xi_{j_1,\dots,j_{n-1-l}}. \tag{3.15}$$

is the irreducible component decomposition of Ξ.

We now consider the irreducible variety $\Xi_{j_1,\dots,j_{n-1-l}}$. Suppose that for the subsequence $1 \le i_1 < \cdots < i_{k-1} \le n-1-l$ we have $j_{i_1} = j_{i_2} = \cdots = j_{i_{k-1}} = U$ and that for the complementary subsequence $i_k < \cdots < i_{n-1-l}$ we have $j_{i_k} = j_{i_{k+1}} = \cdots = j_{i_{n-1-l}} = L$. Then a simple computation with flags shows that elements of the variety $\Xi_{j_1,\dots,j_{n-1-l}}$ stabilize the $n-l$-step partial flag in $\mathbb{C}^n$

$$e_{i_1} \subset e_{i_2} \subset \cdots \subset e_{i_{k-1}} \subset \underbrace{e_{n-l},\dots,e_{n-1},e_n}_{k} \subset e_{i_k} \subset e_{i_{k+1}} \subset \cdots \subset e_{i_{n-1-l}}. \tag{3.16}$$

(If $l=0$ the partial flag in (3.16) is a full flag with e_n in the k-th position.) It is easy to see that there is an element of K that maps the partial flag in Equation (3.16) to the partial flag $\mathcal{P}_{k,k+l}$ in Equation (2.14):

$$\mathcal{P}_{k,k+l} = (e_1 \subset e_2 \subset \cdots \subset e_{k-1} \subset \underbrace{e_k,\dots,e_{k+l-1},e_n}_{k} \subset e_{k+l} \subset \cdots \subset e_{n-1}). \tag{3.17}$$

(If $l=0$ the partial flag $\mathcal{P}_{k,k+l}$ is the full flag $\mathcal{F}_{k,k}$ (see Equation (2.8)).) Thus, $\Xi_{j_1,\dots,j_{n-1-l}} \subset Y_{\mathfrak{p}_{k,k+l}} \cap \mathfrak{g}(n-1-l)$. Equation (3.15) then implies that $\Xi \subset \bigcup_{j-i=l} Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l)$.

Q.E.D.

Using Theorem 3.7, we can obtain the irreducible component decomposition of the variety $\mathfrak{g}(\ge n-1-l)$ for any $l = 0,\dots,n-1$.

Corollary 3.8. *The irreducible component decomposition of the variety* $\mathfrak{g}(\ge n-1-l)$ *is*

$$\mathfrak{g}(\ge n-1-l) = \bigcup_{j-i=l} Y_{\mathfrak{p}_{i,j}} = \bigcup_{l(Q)=l} \overline{Y_Q}. \tag{3.18}$$

Proof. Taking Zariski closures in Equation (3.6), we obtain

$$\overline{\mathfrak{g}(n-1-l)} = \bigcup_{j-i=l} \overline{Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l)} \tag{3.19}$$

is the irreducible component decomposition of the variety $\overline{\mathfrak{g}(n-1-l)}$. By Proposition 3.5, $\overline{\mathfrak{g}(n-1-l)} = \mathfrak{g}(\ge n-1-l)$, and by Theorem 3.6, $Y_{\mathfrak{p}_{i,j}} \subset \mathfrak{g}(\ge n-1-l)$. Hence $Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l)$ is Zariski open in the irreducible variety $Y_{\mathfrak{p}_{i,j}}$, and is nonempty by Theorem 3.7. Therefore $\overline{Y_{\mathfrak{p}_{i,j}} \cap \mathfrak{g}(n-1-l)} = Y_{\mathfrak{p}_{i,j}}$. Equation (3.18) now follows from Equation (3.19) and Proposition 2.12.

Q.E.D.

Theorem 3.7 says something of particular interest to linear algebraists in the case where $l = 0$. It states that the variety $\mathfrak{g}(n-1)$ consisting of elements $x \in \mathfrak{g}$ where the number of coincidences in the spectrum between x_{n-1} and x is maximal can be described in terms of closed K-orbits on $\mathcal{B}$, which are the K-orbits Q with $l(Q) = 0$. It thus connects the most degenerate case of spectral coincidences to the simplest K-orbits on $\mathcal{B}$. More precisely, we have:

Corollary 3.9. *The irreducible component decomposition of the variety* $\mathfrak{g}(n-1)$ *is*

$$\mathfrak{g}(n-1) = \bigcup_{l(Q)=0} Y_Q.$$

Using Corollary 3.9 and Theorem 2.2, we obtain a precise description of the irreducible components of the variety SN_n introduced in Equation (2.4).

Proposition 3.10. *Let* $\mathfrak{b}_{i,i}$ *be the Borel subalgebra of* $\mathfrak{g}$ *which stabilizes the flag* $\mathcal{F}_{i,i}$ *in Equation (2.8) and let* $\mathfrak{n}_{i,i} = [\mathfrak{b}_{i,i}, \mathfrak{b}_{i,i}]$*. The irreducible component decomposition of* SN_n *is given by:*

$$SN_n = \bigcup_{i=1}^{n} \mathrm{Ad}(K)\mathfrak{n}_{i,i}, \tag{3.20}$$

where $\mathrm{Ad}(K)\mathfrak{n}_{i,i} \subset \mathfrak{g}$ *denotes the* K*-saturation of* $\mathfrak{n}_{i,i}$ *in* $\mathfrak{g}$*.*

Proof. We first show that $\mathrm{Ad}(K)\mathfrak{n}_{i,i}$ is an irreducible component of SN_n for $i = 1, \dots, n$. A simple computation using the flag $\mathcal{F}_{i,i}$ in Equation (2.8) shows that $\mathfrak{n}_{i,i} \subset SN_n$. Since SN_n is K-stable, it follows that $\mathrm{Ad}(K)\mathfrak{n}_{i,i} \subset SN_n$.

Recall the Grothendieck resolution $\widetilde{\mathfrak{g}} = \{(x, \mathfrak{b}) : x \in \mathfrak{b}\} \subset \mathfrak{g} \times \mathcal{B}$ and the morphisms $\pi : \widetilde{\mathfrak{g}} \to \mathcal{B}$, $\pi(x, \mathfrak{b}) = \mathfrak{b}$ and $\mu : \widetilde{\mathfrak{g}} \to \mathfrak{g}$, $\mu(x, \mathfrak{b}) = x$. Let $Q_{i,i} = K \cdot \mathfrak{b}_{i,i} \subset \mathcal{B}$ be the K-orbit through $\mathfrak{b}_{i,i}$. Corollary 3.1.33 of [CG97] gives a G-equivariant isomorphism $\widetilde{\mathfrak{g}} \cong G \times_{B_{i,i}} \mathfrak{b}_{i,i}$. Under this isomorphism $\pi^{-1}(Q_{i,i})$ is identified with the closed subvariety $K \times_{K\cap B_{i,i}} \mathfrak{b}_{i,i} \subset G \times_{B_{i,i}} \mathfrak{b}_{i,i}$. The closed subvariety $K \times_{K\cap B_{i,i}} \mathfrak{n}_{i,i} \subset K \times_{K\cap B_{i,i}} \mathfrak{b}_{i,i}$ maps surjectively under μ to $\mathrm{Ad}(K)\mathfrak{n}_{i,i}$. Since μ is proper, $\mathrm{Ad}(K)\mathfrak{n}_{i,i}$ is closed and irreducible. We also note that the restriction of μ to $K \times_{K\cap B_{i,i}} \mathfrak{n}_{i,i}$ generically has finite fibers (Proposition 3.2.14 of [CG97]). Thus, the same reasoning that we used in Equation (2.19) shows that

$$\dim \mathrm{Ad}(K)\mathfrak{n}_{i,i} = \dim(K \times_{K\cap B_{i,i}} \mathfrak{n}_{i,i}) = \dim(Y_{Q_{i,i}}) - \mathrm{rk}(\mathfrak{g}) = d_n, \tag{3.21}$$

where $\mathrm{rk}(\mathfrak{g})$ denotes the rank of $\mathfrak{g}$. Thus, by Theorem 2.2, $\mathrm{Ad}(K)\mathfrak{n}_{i,i}$ is an irreducible component of SN_n.

We now show that every irreducible component of SN_n is of the form $\mathrm{Ad}(K)\mathfrak{n}_{i,i}$ for some $i = 1, \dots, n$. It follows from definitions that $SN_n \subset \mathfrak{g}(n-1) \cap \mathcal{N}$, where $\mathcal{N} \subset \mathfrak{g}$ is the nilpotent cone in $\mathfrak{g}$. Thus, if $\mathfrak{X}$ is an irreducible component of SN_n, then $\mathfrak{X} \subset \mathrm{Ad}(K)\mathfrak{n}_{i,i}$ by Corollary 3.9. But then $\mathfrak{X} = \mathrm{Ad}(K)\mathfrak{n}_{i,i}$ by Equation (3.21) and Theorem 2.2.

Q.E.D.

We say that an element $x \in \mathfrak{g}$ is *n-strongly regular* if the set

$$dJZ_n(x) := \{df_{i,j}(x) : i = n-1, n; j = 1, \ldots, i\}$$

is linearly independent in the cotangent space $T_x^*(\mathfrak{g})$ of $\mathfrak{g}$ at x. We view $\mathfrak{g}_{n-1}$ as the top lefthand corner of $\mathfrak{g}$. It follows from a well-known result of Kostant (Theorem 9 of [Kos63]) that $x_i \in \mathfrak{g}_i$ is regular if and only if the set $\{df_{i,j}(x) : j = 1, \ldots, i\}$ is linearly independent. If $x_i \in \mathfrak{g}_i$ is regular, and we identify $T_x^*(\mathfrak{g}) = \mathfrak{g}^*$ with $\mathfrak{g}$ using the trace form $<< x, y >>= tr(xy)$, then

$$\text{span } \{df_{i,j}(x) : j = 1, \ldots, i\} = \mathfrak{z}_{\mathfrak{g}_i}(x_i),$$

where $\mathfrak{z}_{\mathfrak{g}_i}(x_i)$ denotes the centralizer of x_i in $\mathfrak{g}_i$. Thus, it follows that $x \in \mathfrak{g}$ is n-strongly regular if and only if x satisfies the following two conditions:

$$\begin{aligned} &(1)\ x \in \mathfrak{g} \text{ and } x_{n-1} \in \mathfrak{g}_{n-1} \text{ are regular; and} \\ &(2)\ \mathfrak{z}_{\mathfrak{g}_{n-1}}(x_{n-1}) \cap \mathfrak{z}_{\mathfrak{g}}(x) = 0. \end{aligned} \tag{3.22}$$

Remark 3.11. *We claim that the ideal I_n is radical if and only if $n \leq 2$. The assertion is clear for $n = 1$, and we assume $n \geq 2$ in the sequel. Indeed, by Theorem 18.15(a) of* [Eis95]*, the ideal I_n is radical if and only if the set dJZ_n is linearly independent on a dense open set of each irreducible component of $SN_n = V(I_n)$. It follows that I_n is radical if and only if each irreducible component of SN_n contains n-strongly regular elements. Let $\mathfrak{n}_+ = [\mathfrak{b}_+, \mathfrak{b}_+]$ and $\mathfrak{n}_- = [\mathfrak{b}_-, \mathfrak{b}_-]$ be the strictly upper and lower triangular matrices, respectively. By Proposition 3.10 above, SN_n has exactly n irreducible components. It follows from the discussion after Equation (2.8) that two of them are $K \cdot \mathfrak{n}_+$ and $K \cdot \mathfrak{n}_-$. By Proposition 3.10 of* [CE12]*, the only irreducible components of SN_n which contain n-strongly regular elements are $K \cdot \mathfrak{n}_+$ and $K \cdot \mathfrak{n}_-$. The claim now follows. See Remark 1.1 of* [Ovs03] *for a related observation, which follows also from the analysis proving our claim.*

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